

Geodesic Particle Paths Inside a Nonrotating, Homogeneous, Spherical Body

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(Dated: October 16, 2012)

Proceeding from a solution of field equations that are improved versions of Einstein's nonvacuum gravitational field equations one is able to calculate precisely the trajectories of particles traveling inside a nonrotating, homogeneous, spherical body. Application of the results to the conditions of recent measurements of neutrino flight times between a source point A at CERN's European Laboratory for Particle Physics and a point B in either of two detectors (ICARUS or OPERA) at LNGS (Laboratori Nazionale del Gran Sasso), separated by a euclidean distance $d(A, B) = 731$ km, predicts for the flight time T_ν from A to B of a 2 eV neutrino launched with energy 17 GeV that, as measured by a clock at B synchronized to a similar clock at A, $T_\nu \approx d/c + 9.3 \times 10^{-16}$ sec. But as measured by inertial observers along the path the flight time $\bar{T}_\nu \approx d/c - 2.6 \times 10^{-9}$ sec and the path length $L_\nu \approx d - 8.4 \times 10^{-7}$ m, which yields $L_\nu/\bar{T}_\nu \approx c + 321$ m/sec for the average inertially referenced speed of the neutrino from A to B.

PACS numbers: 04.50.Kd, 04.20.Jb, 04.40.Nr

I. THE INTERIOR METRIC OF THE SPHERICAL BODY

In a previous paper I derived a space-time metric for the gravitational field inside a nonrotating, homogeneous, spherical ball \mathcal{B} , matched as smoothly as possible at the surface of \mathcal{B} to a Schwarzschild exterior metric [1]. This new metric is a solution of the field equations

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -\frac{4\pi\kappa}{c^2}\mu g_{\alpha\beta}, \quad (1)$$

which come from the variational principle

$$\delta \int \left(R - \frac{8\pi\kappa}{c^2}\mu \right) |g|^{\frac{1}{2}} d^4x = 0, \quad (2)$$

in which κ is Newton's gravitational constant and μ is *active* gravitational mass density, and which is the most natural extension to the general relativity setting of the variational principle $\delta \int (|\nabla V|^2 + 8\pi\kappa\mu V) d^3x = 0$ that produces the Poisson equation $\nabla^2 V = 4\pi\kappa\mu$ for the newtonian gravitational potential V .¹ Because these field equations differ from those that Einstein postulated in 1916 ([4], §16, §19) (and that have been taken as gospel ever since), the interior solution they yielded in [1] differs significantly from the Schwarzschild interior solution [1, 5]. In particular, the metric can be expressed entirely in terms of rational functions of the radial coordinate, which makes feasible a relatively straightforward analysis of its geodesics and thereby allows computations of flight times and travel distances of test particles such as photons and neutrinos following geodesics between points on the surface of the spherical ball considered as representing Earth (provided with tunnels for the photons to travel in affected only by gravity).

The metric takes the proper-time forms

$$d\tau^2 = [1 - f^2(\rho)] dt^2 - \frac{1}{c^2} [1 - f^2(\rho)]^{-1} d\rho^2 - \frac{1}{c^2} r^2(\rho) d\Omega^2 \quad (3)$$

$$= d\bar{t}^2 - \frac{1}{c^2} [d\rho - f(\rho) c d\bar{t}]^2 - \frac{1}{c^2} r^2(\rho) d\Omega^2, \quad (4)$$

in which $\bar{t} := t - (1/c) \int f(\rho) [1 - f^2(\rho)]^{-1} d\rho$, $r(\rho) = \lambda(\rho - \rho_0)$ (the areal radius of a sphere of geodesic radius $\rho - \rho_0$ in a generic \bar{t} time slice, normalized so that $r(\rho) = R$ when $\rho = R$, the radius of the ball \mathcal{B} , which makes $\rho_0 = (1 - 1/\lambda) R$),

$$1 - f^2 = \frac{1}{\lambda^2} \left(1 + \frac{\lambda\kappa M}{c^2 R} \frac{r^2}{R^2} \right) = \frac{1}{\lambda^2} \left(1 + \frac{\lambda m}{R^3} r^2 \right), \quad (5)$$

$f := -\sqrt{f^2}$, M is the active gravitational mass of \mathcal{B} , $m = \kappa M/c^2$ ($= M$ in geometric units), and the dimensionless parameter

$$\lambda = \frac{m + \sqrt{m^2 + 4R(R - 2m)}}{2(R - 2m)}. \quad (6)$$

¹ Justification for the complete version of this variational principle and the field equations that it implies can be seen in [2] and, in greater detail, in [3].

It is assumed that $R > 2m$ (the Schwarzschild radius of \mathcal{B}), from which it follows that $\lambda > 1$ and $\rho_0 > 0$.

The vector field $\partial_{\bar{t}} + f(\rho)c\partial_\rho$ is the velocity field of a cloud of inertial observers free-falling from rest at $\rho = \infty$; the time \bar{t} runs at the same rate as their proper times. The geometry of space as seen by these observers is described by the metric of a time-slice $\Sigma_{\bar{t}}$ of constant \bar{t} , namely $d\sigma^2 = d\rho^2 + r^2(\rho)d\Omega^2$. Rather than being flat euclidean, as in a corresponding slice of the Schwarzschild exterior metric, in which $r(\rho) = \rho$ (the geodesic distance from the point singularity where r would be 0 if the ball \mathcal{B} were collapsed to that point), they are ‘hyperconical’ in that $r(\rho) = \lambda(\rho - \rho_0) > \rho - \rho_0$ (the geodesic distance from the center of \mathcal{B} where $\rho = \rho_0$ and r is in fact 0 — the vertex of the ‘hypercone’). Although $\Sigma_{\bar{t}}$ has a curvature singularity at the vertex of the hypercone (the $\vartheta\varphi$ sectional curvature being $(1 - r'^2)/r^2 = -(\lambda^2 - 1)/r^2$), the full space-time manifold does not, as one can show that inside \mathcal{B} every one of the sectional curvatures is the same $-\lambda m/R^3$.

II. GEODESICS INSIDE THE SPHERICAL BODY \mathcal{B}

To study geodesics inside the spherical ball \mathcal{B} , in particular geodesics between two points A and B on the surface of \mathcal{B} , let us orient the spherical polar coordinate system of Eqs. (3) and (4) so that A and B are on a longitude and equidistant from the equator (A to the north and B to the south), and in place of the usual colatitude coordinate ϑ use the latitude coordinate θ , the two related by $\theta = \pi/2 - \vartheta$. Then $d\Omega^2 = d\vartheta^2 + (\sin \vartheta)^2 d\varphi^2 = d\theta^2 + (\cos \theta)^2 d\varphi^2$, and $\theta = \delta$ at A and $-\delta$ at B, where $\delta = \sin^{-1}(d/2R)$ and d is the euclidean distance from A to B.

For every affinely parametrized geodesic path in \mathcal{B} with longitude φ fixed there are three additional constants of the motion, namely,

$$h := \frac{1}{c} r^2 \dot{\vartheta} = -\frac{1}{c} r^2 \dot{\theta}, \quad (7)$$

$$k := (1 - f^2) \dot{t} = \dot{t} + \frac{f}{c} (\dot{\rho} - f c \dot{t}), \quad (8)$$

and

$$\epsilon := (1 - f^2) \dot{t}^2 - \frac{1}{c^2} \frac{1}{1 - f^2} \dot{\rho}^2 - \frac{1}{c^2} r^2 \dot{\theta}^2, \quad (9)$$

$$= \dot{t}^2 - \frac{1}{c^2} (\dot{\rho} - f c \dot{t})^2 - \frac{1}{c^2} r^2 \dot{\theta}^2, \quad (10)$$

where $\epsilon = 1, 0, -1$ according as the path is timelike (parametrized by arclength), lightlike, or spacelike (parametrized by arclength). From these equations and Eq. (5) follow

$$\dot{\rho}^2 = c^2 \left[k^2 - (1 - f^2) \left(\frac{h^2}{r^2} + \epsilon \right) \right] \quad (11)$$

$$= -\frac{c^2}{r^2} \left[\epsilon \frac{m}{\lambda R^3} r^4 - \frac{(\lambda^2 k^2 - \epsilon) R^3 - \lambda m h^2}{\lambda^2 R^3} r^2 + \frac{h^2}{\lambda^2} \right]. \quad (12)$$

A. Lightlike geodesics

For a lightlike geodesic $\epsilon = 0$ and Eq. (12) reduces to

$$\dot{\rho}^2 = \frac{c^2}{r^2} \left[\frac{\lambda^2 k^2 R^3 - \lambda m h^2}{\lambda^2 R^3} r^2 - \frac{h^2}{\lambda^2} \right]. \quad (13)$$

If $\lambda^2 k^2 R^3 - \lambda m h^2 \leq 0$, then nonnegativity of $\dot{\rho}^2$ forces $h = 0$, which then forces $k = 0$; the geodesic is degenerate, comprising a single event. If $\lambda^2 k^2 R^3 - \lambda m h^2 > 0$, then either $h = 0$, in which case $\dot{\theta} = 0$ and the geodesic traces out a diameter of \mathcal{B} , or else $h \neq 0$ and Eq. (13), which precludes $k = 0$, when combined with Eq. (7) yields

$$\left(\frac{dr}{d\theta} \right)^2 = \left(\frac{dr}{d\rho} \right)^2 \left(\frac{d\rho}{d\theta} \right)^2 = \lambda^2 \frac{\dot{\rho}^2}{\dot{\theta}^2} = \frac{r^2 (r^2 - r_0^2)}{r_0^2}, \quad (14)$$

where

$$r_0 = \sqrt{\frac{R^3 (h/k)^2}{\lambda^2 R^3 - \lambda m (h/k)^2}}. \quad (15)$$

As a lightlike particle travels from A to B (through a tunnel created for its passage) the value of r decreases from R at A to r_0 at the halfway point H, where $\theta = 0$, then increases back to R at B. From A to H, when θ is positive, $d\theta/dr > 0$, and from H to B, when θ is negative, $d\theta/dr < 0$, so

$$\theta = \text{sgn}(\theta) \int_{r_0}^r \frac{d\theta}{ds} ds = \text{sgn}(\theta) \int_{r_0}^r \frac{r_0}{s \sqrt{s^2 - r_0^2}} ds = \text{sgn}(\theta) \sec^{-1} \left(\frac{r}{r_0} \right), \quad (16)$$

and therefore $r = r_0 \sec(\text{sgn}(\theta) \theta) = r_0 \sec \theta$. This describes a trajectory that in euclidean geometry, where $x = r \cos \theta$ and $y = r \sin \theta$, would be a straight line interval from A to B. To determine r_0 , note that, at A, $r = R$ and $\theta = \delta$, so that $R = r_0 \sec(\delta)$, thus $r_0 = R \cos(\delta) = R \cos(\sin^{-1}(d/2R)) = \sqrt{R^2 - (d/2)^2}$. The geodesic distance from the center \mathcal{C} of \mathcal{B} to the halfway point H is then $R \cos(\delta)/\lambda$.

From Eqs. (9), (14), and (5), together with $r = r_0 \sec \theta$, it is relatively straightforward to calculate that

$$\left(\frac{dt}{d\theta} \right)^2 = \frac{\lambda^2 r_0^2 [1 + (\lambda m/R^3) r_0^2]}{c^2 [(\cos \theta)^2 + (\lambda m/R^3) r_0^2]^2}. \quad (17)$$

If the particle starts from A at time t_A and arrives at B at time t_B , then, because t is increasing as θ is decreasing,

$$t_B - t_A = \int_{\delta}^{-\delta} \frac{dt}{d\theta} d\theta = \frac{1}{c} \int_{-\delta}^{\delta} \frac{\lambda r_0 \sqrt{1 + (\lambda m/R^3) r_0^2}}{(\cos \theta)^2 + (\lambda m/R^3) r_0^2} d\theta \quad (18)$$

$$= \frac{2}{c} \sqrt{\frac{\lambda R^3}{m}} \tan^{-1} \left(\sqrt{\frac{\lambda m r_0^2}{R^3 + \lambda m r_0^2}} \tan(\delta) \right) \quad (19)$$

$$= \frac{2}{c} \sqrt{\frac{\lambda R^3}{m}} \tan^{-1} \left(\sqrt{\frac{\lambda m}{R + \lambda m \cos^2(\delta)}} \sin(\delta) \right). \quad (20)$$

An accurate clock at B perfectly synchronized with a matching clock at A would record the flight time $T_{\epsilon=0}$ of the particle as the proper time elapsed at B since the particle left A, that is to say, $T_{\epsilon=0} = \sqrt{1 - f^2(R)} (t_B - t_A)$.

In a similar manner, starting from the metric $d\sigma^2 = d\rho^2 + r^2(\rho) d\Omega^2$ of $\Sigma_{\bar{t}}$, one finds that

$$\left(\frac{d\sigma}{d\theta} \right)^2 = \frac{r_0^2 [\lambda^2 + (\tan \theta)^2]}{\lambda^2 (\cos \theta)^2}, \quad (21)$$

and then that the length $L_{\epsilon=0}$ of the path followed by the particle is given by

$$L_{\epsilon=0} = \frac{R \cos(\delta)}{\lambda} \int_{-\delta}^{\delta} \frac{\sqrt{\lambda^2 + (\tan \theta)^2}}{\cos \theta} d\theta, \quad (22)$$

which involves elliptic integrals so must be integrated numerically. It is straightforward to show analytically that the result will lie between d/λ and d .

To find the (average) speed of a photon on a flight from A to B, as measured by inertial observers free-falling from rest at $\rho = \infty$, we need $\bar{t}_B - \bar{t}_A$. From $\bar{t} := t - (1/c) \int f(\rho) [1 - f^2(\rho)]^{-1} d\rho = t - (1/\lambda c) \int f(1 - f^2)^{-1} dr = t - (1/\lambda c) \int f(1 - f^2)^{-1} (dr/d\theta) d\theta$, (anti)symmetry between the flight from A to H and the flight from H to B, Eq. (5), and $f := -\sqrt{f^2}$, we get that

$$\bar{T}_{\epsilon=0} := \bar{t}_B - \bar{t}_A = t_B - t_A - \frac{1}{\lambda c} \int_{\delta}^{-\delta} \frac{f}{1 - f^2} \frac{dr}{d\theta} d\theta = t_B - t_A + \frac{1}{\lambda c} \int_{-\delta}^{\delta} \frac{f}{1 - f^2} \frac{dr}{d\theta} d\theta \quad (23)$$

$$= t_B - t_A + \frac{2}{\lambda c} \int_0^{\delta} \frac{f}{1 - f^2} \frac{dr}{d\theta} d\theta = t_B - t_A + \frac{2}{\lambda c} \int_{r_0}^R \frac{f}{1 - f^2} dr \quad (24)$$

$$= t_B - t_A - \frac{2}{c} \int_{r_0}^R \frac{\sqrt{\lambda^2 - 1 - (\lambda m/R^3) r^2}}{1 + (\lambda m/R^3) r^2} dr. \quad (25)$$

Application: Photon flight from CERN to Gran Sasso

These results can be immediately applied to a situation of recent interest by choosing for d the euclidean distance between the end points A (CERN) and B (Gran Sasso) of the neutrino flight path in the experiments described in [6] and [7], that is, $d = 731$ km (rounded). With $M = 5.9722 \times 10^{24}$ kg (the active gravitational mass of Earth) and $R = 6.3710 \times 10^3$ km (the mean radius of Earth) the outcomes, compared to d , c , and $d/c = 0.00244$ sec, are

- photon's flight time measured by clock at B synchronized with clock at A:

$$T_{\epsilon=0} = d/c + 9.31085 \times 10^{-16} \text{ sec} \quad (26)$$

- photon's flight time from A to B as measured by free-falling inertial observers:

$$\bar{T}_{\epsilon=0} = d/c - 2.61131 \times 10^{-9} \text{ sec} \quad (27)$$

- length of photon's flight path from A to B as measured by free-falling inertial observers:

$$L_{\epsilon=0} = d - 8.38501 \times 10^{-5} \text{ cm} = d - 0.838501 \mu\text{m} \quad (28)$$

- average speed of photon in flight from A to B as measured by free-falling inertial observers:

$$\frac{L_{\epsilon=0}}{\bar{T}_{\epsilon=0}} = c + 0.32106 \text{ km/sec} \quad (= 1.00000107093 c = (1 + 1.07093 \times 10^{-6}) c) \quad (29)$$

The corresponding numbers for neutrino flights from A to B will be found in the next section.

B. Timelike geodesics

For a timelike geodesic parametrized by the arclength parameter τ , $\epsilon = 1$ and Eq. (12) reduces to

$$\dot{\rho}^2 = -\frac{c^2 m}{\lambda R^3 r^2} \left(r^4 - 2\alpha r^2 + \frac{R^3 h^2}{\lambda m} \right), \quad (30)$$

where

$$\alpha = \frac{(\lambda^2 k^2 - 1)R^3 - \lambda m h^2}{2\lambda m}. \quad (31)$$

If $\alpha \leq 0$, then nonnegativity of $\dot{\rho}^2$ forces $h = 0$ and $r = 0$, so a particle on this path would be forever stuck at the center \mathcal{C} of \mathcal{B} . If $\alpha > 0$, then Eq. (30) becomes

$$\dot{\rho}^2 = \frac{c^2 m}{\lambda R^3 r^2} (r^2 - a^2)(b^2 - r^2), \quad (32)$$

where $a = \sqrt{\alpha - \beta}$, $b = \sqrt{\alpha + \beta}$, and $\beta = \sqrt{\alpha^2 - R^3 h^2 / \lambda m}$. If $h = 0$, then $\dot{\theta} = 0$ and $a = 0$, and the particle's position oscillates along a diameter of \mathcal{B} between extremes at $r = b$, if $b \leq R$, or else enters \mathcal{B} at one end of a diameter and exits at the other end (in either case, $r = b |\sin(\sqrt{\lambda m / R^3} c \tau)|$). If $h \neq 0$, then $a > 0$ and Eq. (32) combined with Eq. (7) produces

$$\left(\frac{dr}{d\theta} \right)^2 = \lambda^2 \frac{\dot{\rho}^2}{\dot{\theta}^2} = \frac{\lambda m}{R^3 h^2} r^2 (r^2 - a^2)(b^2 - r^2) = \frac{r^2 (r^2 - a^2)(b^2 - r^2)}{a^2 b^2}, \quad (33)$$

This implies that $a \leq r \leq b$, and if $r = a$ when $\theta = 0$, then, as in Eq. (16),

$$\theta = \text{sgn}(\theta) \int_a^r \frac{d\theta}{ds} ds = \text{sgn}(\theta) \int_a^r \frac{ab}{s \sqrt{(b^2 - s^2)(s^2 - a^2)}} ds \quad (34)$$

$$= \text{sgn}(\theta) \tan^{-1} \left(\frac{b}{a} \sqrt{\frac{r^2 - a^2}{b^2 - r^2}} \right), \quad (35)$$

from which follows

$$r = \frac{a b}{\sqrt{a^2 (\sin \theta)^2 + b^2 (\cos \theta)^2}}. \quad (36)$$

If $b \leq R$, this describes an oval orbit within \mathcal{B} that in euclidean geometry, where $x = r \cos \theta$ and $y = r \sin \theta$, would be an ellipse centered on \mathcal{C} , with minor (x) axis of length $2a$ and major (y) axis of length $2b$. If $b > R$, the trajectory described is an arc of such an oval connecting points A and B on the surface of \mathcal{B} at which $r = R$ and $\theta = \pm \delta = \pm \sin^{-1}(d/2R)$, where d is the euclidean distance from A to B. Used in Eq. (36), $r = R$ and $\theta = \delta$ produce

$$a b = R \sqrt{a^2 (\sin \delta)^2 + b^2 (\cos \delta)^2} \quad (37)$$

as an initial condition to help fix the constants h and k . A useful consequence of this equation and $a^2 b^2 = \alpha^2 - \beta^2 = R^3 h^2 / \lambda m$ is that

$$h^2 = \frac{\lambda m}{R^3} a^4 \frac{R^2 - r_0^2}{a^2 - r_0^2}, \quad (38)$$

from which follows that $a > r_0$ and $h \rightarrow \infty$ as $a \rightarrow r_0$ (and vice versa).

From $d\theta/d\tau = \dot{\theta} = -c h / r^2$ and Eq. (36) one can find the proper time elapsed on the particle's clock as follows:

$$\tau_B - \tau_A = \int_{\tau_A}^{\tau_B} d\tau = -\frac{1}{c h} \int_{\delta}^{-\delta} r^2 d\theta = \frac{a^2 b^2}{c h} \int_{-\delta}^{\delta} \frac{1}{a^2 (\sin \theta)^2 + b^2 (\cos \theta)^2} d\theta \quad (39)$$

$$= \frac{2 a b}{c h} \tan^{-1} \left(\frac{a}{b} \tan(\delta) \right). \quad (40)$$

To find the proper time elapsed on a clock at B synchronized with a clock at A requires computing $t_B - t_A$ as follows:

$$\left(\frac{d\tau}{d\theta} \right)^2 = (1 - f^2) \left(\frac{dt}{d\theta} \right)^2 - \frac{1}{c^2} \left[\frac{1}{\lambda^2 (1 - f^2)} \left(\frac{dr}{d\theta} \right)^2 + r^2 \right], \quad (41)$$

so

$$\left(\frac{dt}{d\theta} \right)^2 = \frac{1}{c^2 (1 - f^2)} \left[c^2 \left(\frac{d\tau}{d\theta} \right)^2 + \frac{1}{\lambda^2 (1 - f^2)} \left(\frac{dr}{d\theta} \right)^2 + r^2 \right] \quad (42)$$

$$= \frac{\lambda^2}{c^2 [\lambda^2 (1 - f^2)]^2} \left\{ \lambda^2 (1 - f^2) \left[c^2 \left(\frac{d\tau}{d\theta} \right)^2 + r^2 \right] + \left(\frac{dr}{d\theta} \right)^2 \right\}. \quad (43)$$

After substitutions from $d\tau/d\theta = -r^2/c h$ and Eqs. (5), (33), and (36) this reduces to

$$\left(\frac{dt}{d\theta} \right)^2 = \frac{\lambda^2 a^2 b^2 (h^2 + a^2)(h^2 + b^2)}{c^2 h^2 [h^2 + a^2 (\sin \theta)^2 + b^2 (\cos \theta)^2]^2}, \quad (44)$$

which yields

$$t_B - t_A = \int_{\delta}^{-\delta} \frac{dt}{d\theta} d\theta = \frac{1}{c h} \int_{-\delta}^{\delta} \frac{\lambda a b \sqrt{(h^2 + a^2)(h^2 + b^2)}}{h^2 + a^2 (\sin \theta)^2 + b^2 (\cos \theta)^2} d\theta \quad (45)$$

$$= \frac{2 \lambda a b}{c h} \tan^{-1} \left(\sqrt{\frac{h^2 + a^2}{h^2 + b^2}} \tan(\delta) \right). \quad (46)$$

The flight time $T_{\epsilon=1}$ of the particle as read on a clock at B perfectly synchronized with a matching clock at A is given by $T_{\epsilon=1} = \sqrt{1 - f^2(R)} (t_B - t_A)$.

From the metric $d\sigma^2 = d\rho^2 + r^2(\rho)d\Omega^2$ of $\Sigma_{\bar{t}}$ and Eq. (33) one gets

$$\left(\frac{d\sigma}{d\theta}\right)^2 = \left(\frac{d\rho}{d\theta}\right)^2 + r^2 = \frac{1}{\lambda^2} \left(\frac{dr}{d\theta}\right)^2 + r^2 \quad (47)$$

$$= \frac{1}{\lambda^2} \frac{r^2(r^2 - a^2)(b^2 - r^2)}{a^2 b^2} + r^2 \quad (48)$$

$$= \frac{r^6}{\lambda^2 a^2 b^2} \left[\left(1 - \frac{a^2}{r^2}\right) \left(\frac{b^2}{r^2} - 1\right) + \frac{\lambda^2 a^2 b^2}{r^4} \right]. \quad (49)$$

Substitution from Eq. (36) produces for the length $L_{\epsilon=1}$ of the particle's path

$$L_{\epsilon=1} = \frac{ab}{\lambda} \int_{-\delta}^{\delta} \sqrt{\frac{a^4(\sin\theta)^2 + (\lambda^2 - 1)[a^2(\sin\theta)^2 + b^2(\cos\theta)^2]^2 + b^4(\cos\theta)^2}{[a^2(\sin\theta)^2 + b^2(\cos\theta)^2]^3}} d\theta. \quad (50)$$

The formula for $\bar{T}_{\epsilon=1}$ is like that of Eq. (25) for $\bar{T}_{\epsilon=0}$, viz.,

$$\bar{T}_{\epsilon=1} := \bar{t}_B - \bar{t}_A = t_B - t_A - \frac{2}{c} \int_a^R \frac{\sqrt{\lambda^2 - 1 - (\lambda m/R^3)r^2}}{1 + (\lambda m/R^3)r^2} dr. \quad (51)$$

Application: Neutrino flight from CERN to Gran Sasso

To apply these results to the flight from A to B of a particle such as a neutrino one needs two equations to determine the constants h and k . One of these will be Eq. (37), the other must involve the particle's rest mass m_0 and the value at A of its energy E , related by the well-known formula $\hat{E} := E/m_0 c^2 = 1/\sqrt{1 - v^2/c^2}$, where at each event on the particle's path v is the magnitude of its coordinate three-velocity with respect to an inertial observer at that event. At every event such an observer \mathcal{O} is one that is falling freely from rest at $\rho = \infty$ with no angular motion, whose coordinate four-velocity is $\partial_{\bar{t}} + f(\rho) c \partial_{\rho}$. The particle's coordinate four-velocity is $\partial_{\bar{t}} + (d\rho/d\bar{t}) \partial_{\rho} + (d\theta/d\bar{t}) \partial_{\theta}$. Their relative coordinate three-velocity is thus $[d\rho/d\bar{t} - f(\rho) c] \partial_{\rho} + (d\theta/d\bar{t}) \partial_{\theta}$, the square of whose magnitude v as measured in the metric of $\Sigma_{\bar{t}}$ (the metric of space as seen by \mathcal{O}) is given by $v^2 = [d\rho/d\bar{t} - f(\rho) c]^2 + r^2(\rho)(d\theta/d\bar{t})^2$. From Eq. (4) one gets

$$1 = \left(\frac{d\bar{t}}{d\tau}\right)^2 - \frac{1}{c^2} \left[\frac{d\rho}{d\tau} - f(\rho) c \frac{d\bar{t}}{d\tau}\right]^2 - \frac{1}{c^2} r^2(\rho) \left(\frac{d\theta}{d\tau}\right)^2 \quad (52)$$

$$= \left(\frac{d\bar{t}}{d\tau}\right)^2 \left\{ 1 - \frac{1}{c^2} \left[\frac{d\rho}{d\bar{t}} - f(\rho) c\right]^2 - \frac{1}{c^2} r^2(\rho) \left(\frac{d\theta}{d\bar{t}}\right)^2 \right\} \quad (53)$$

$$= \left(\frac{d\bar{t}}{d\tau}\right)^2 \left(1 - \frac{v^2}{c^2}\right), \quad (54)$$

so $\hat{E} = |d\bar{t}/d\tau| = |\dot{\bar{t}}| = |\dot{t} - (1/c)f(\rho)[1 - f^2(\rho)]^{-1}\dot{\rho}|$. From Eq. (52) follows $[\dot{\rho} - f(\rho) c \dot{\bar{t}}]^2 = c^2 \dot{\bar{t}}^2 - r^2(\rho) \dot{\theta}^2 - c^2$, and then from Eq. (7)

$$\dot{\rho} - f(\rho) c \dot{\bar{t}} = \sqrt{c^2 \dot{\bar{t}}^2 - r^2(\rho) \dot{\theta}^2 - c^2} = c \sqrt{\hat{E}^2 - \frac{h^2}{r^2(\rho)} - 1}, \quad (55)$$

where the positive root is chosen to account for the fact that as time goes on ($\dot{\bar{t}} > 0$) the particle descends into \mathcal{B} more slowly than does the free-falling observer \mathcal{O} ($f(\rho) c < d\rho/d\bar{t} < 0$). Now Eq. (8) gives $k = \hat{E} + f(\rho) \sqrt{\hat{E}^2 - h^2/r^2(\rho) - 1}$, which evaluated at A becomes

$$k = \hat{E}_0 + f(R) \sqrt{\hat{E}_0^2 - \frac{h^2}{R^2} - 1}, \quad (56)$$

where $\hat{E}_0 = E_0/m_0c^2$, the ratio of the initial energy of the particle to its rest energy. Solution of Eqs. (37) and (56) for h and k will enable computation of $\tau_B - \tau_A$, $t_B - t_A$, $T_{\epsilon=1}$, $L_{\epsilon=1}$, and $\bar{T}_{\epsilon=1}$ for various choices of \hat{E}_0 .

Squaring both sides of Eq. (37) produces $\alpha^2 - \beta^2 = R^2\alpha + R^2\beta \sin(2\delta)$ (from $a = \sqrt{\alpha - \beta}$ and $b = \sqrt{\alpha + \beta}$). Transposing the term $R^2\alpha$ and squaring again one arrives ultimately at

$$[4R(R + \lambda m) + \lambda^2 m^2 \sin^2(2\delta)] h^4 - 2R^3 \{(\lambda^2 k^2 - 1)[2R + \lambda m \sin^2(2\delta)] - 2\lambda m \cos^2(2\delta)\} h^2 + R^6 (\lambda^2 k^2 - 1)^2 \sin^2(2\delta) = 0. \quad (57)$$

A similar treatment of Eq. (56) produces

$$[\lambda^2 R - (R + \lambda m)] h^2 + R^2 \left[\lambda^2 R (k^2 - 2\hat{E}_0 k + 1) + (R + \lambda m) (\hat{E}_0^2 - 1) \right] = 0. \quad (58)$$

Substitution of h^2 from the second of these equations into the first produces a polynomial equation of degree four in k , so numerical solution is advised.

In the experiments described in [6] and [7] neutrinos are collected at point(s) B (Gran Sasso), having been launched from point A (CERN) with energy $E_0 \approx 17$ GeV. The euclidean distance $d = 731$ km (rounded) from A to B determined from satellite and ground measurements, the only unknown datum is the neutrino rest energy. An upper bound on this energy is thought to be 2 eV. Taking this for m_0c^2 makes $\hat{E}_0 = 8.5 \times 10^9$ (thus the neutrino initial speed $v_0 = c \sqrt{1 - 1/\hat{E}_0^2} = (1 - 7. \times 10^{-21})c$), for which choice $k = 8.49998 \times 10^9$ and $h = 5.40642 \times 10^{18}$ cm. Use of these in the formulas above gives

- neutrino's proper time elapsed in flight from A to B:

$$\tau_B - \tau_A = 2.86866 \times 10^{-13} \text{ sec} \quad (59)$$

- neutrino's flight time measured by clock at B synchronized with clock at A:

$$T_{\epsilon=1} = d/c + 9.31085 \times 10^{-16} \text{ sec} \quad (= T_{\epsilon=0} + 1.68745 \times 10^{-23} \text{ sec}) \quad (60)$$

- neutrino's flight time from A to B as measured by free-falling inertial observers:

$$\bar{T}_{\epsilon=1} = d/c - 2.61131 \times 10^{-9} \text{ sec} \quad (= \bar{T}_{\epsilon=0} + 1.68745 \times 10^{-23} \text{ sec}) \quad (61)$$

- length of neutrino's flight path from A to B as measured by free-falling inertial observers:

$$L_{\epsilon=1} = d - 8.38501 \times 10^{-5} \text{ cm} = d - 0.838501 \mu\text{m} \quad (= L_{\epsilon=0} + 1.61152 \times 10^{-33} \text{ cm}) \quad (62)$$

- average speed of neutrino in flight from A to B as measured by free-falling inertial observers:

$$\frac{L_{\epsilon=1}}{\bar{T}_{\epsilon=1}} = c + 0.32106 \text{ km/sec} = 1.00000107093 c = (1 + 1.07093 \times 10^{-6}) c \quad \left(= \frac{L_{\epsilon=0}}{\bar{T}_{\epsilon=0}} - 2.07470 \times 10^{-10} \text{ cm/sec} \right) \quad (63)$$

That $T_{\epsilon=1}$ exceeds d/c by 9.31085×10^{-16} sec is the result of relevance to the experiments described in [6] and [7].

Application: Newton's cannonball

Isaac Newton imagined a cannon firing a cannonball horizontally from a high mountaintop with velocity just sufficient to keep it from ever falling to ground. If we bring his cannon down to a point A on the ball \mathcal{B} (taken to represent a nonrotating, homogeneous, spherical Earth along whose surface the cannonball can travel without hindrance), then the formulas derived above will apply with $a = b = R$, in which case we have that $r(\rho) = \rho = R$,

- that $\alpha - \beta = a^2 = R^2$ and $\alpha + \beta = b^2 = R^2$, thus $\alpha = R^2$ and $0 = \beta = \sqrt{\alpha^2 - R^3 h^2 / \lambda m} = \sqrt{R^4 - R^3 h^2 / \lambda m}$, and therefore $h = \sqrt{\lambda m R} = 1.68093 \times 10^4$ cm,
- from Eqs. (7), (8), and (9) that $k^2 = (1 - f^2(R))(1 + h^2/R^2)$, thus that $k = (1/\lambda)(1 + \lambda m/R) = 0.9999999997 = 1 - 3. \times 10^{-10}$, and

- from $k = \hat{E} + f(R)\sqrt{\hat{E}^2 - h^2/R^2 - 1}$ and Eq. (56) that $\hat{E} = \hat{E}_0 = \lambda = 1.000000001 = 1 + 1. \times 10^{-9}$.

These give, as measured by free-falling inertial observers, a flight time for the cannonball's 'round' trip from A to A of 84.34771 min = 1 hr 24 min 20.86261 sec, a flight path length of 4.00302×10^4 km = $2\pi R$, and an average (in fact, a constant) speed of 7.90975 km/sec. For a 12 lb cannonball of inertial rest mass $m_0 = 12$ lb/g in common use in Newton's day the amount of gunpowder required to send it on its way with the required kinetic energy (= total energy - rest energy = $E_0 - m_0 c^2 = (\hat{E}_0 - 1)m_0 c^2 = 3.18861 \times 10^{18}$ GeV = 3.76758×10^8 ft-lb) would be about 377 pounds (at 500 ft-tons per pound of powder [9]).

C. Spacelike geodesics

For a spacelike geodesic parametrized by the arclength parameter $\hat{\tau} := i\tau$, $\epsilon = -1$ and Eq. (12) reduces to

$$\dot{\rho}^2 = \frac{c^2 m}{\lambda R^3 r^2} \left(r^4 + 2\bar{\alpha} r^2 - \frac{R^3 h^2}{\lambda m} \right), \quad (64)$$

where

$$\bar{\alpha} = \frac{(\lambda^2 k^2 + 1)R^3 - \lambda m h^2}{2\lambda m}. \quad (65)$$

If $h = 0$, then $\dot{\theta} = 0$ and the geodesic follows a diameter of \mathcal{B} from one end to the other. If $h \neq 0$, then Eq. (64) combined with Eq. (7) produces

$$\left(\frac{dr}{d\theta} \right)^2 = \lambda^2 \frac{\dot{\rho}^2}{\dot{\theta}^2} = \frac{\lambda m}{R^3 h^2} r^2 (r^2 - \bar{a}^2)(r^2 + \bar{b}^2) = \frac{r^2 (r^2 - \bar{a}^2)(r^2 + \bar{b}^2)}{\bar{a}^2 \bar{b}^2}, \quad (66)$$

where $\bar{a} = \sqrt{\beta - \bar{\alpha}}$, $\bar{b} = \sqrt{\beta + \bar{\alpha}}$, and $\bar{\beta} = \sqrt{\bar{\alpha}^2 + R^3 h^2 / \lambda m}$. This implies that $0 < \bar{a} \leq r$, and if $r = \bar{a}$ when $\theta = 0$, then, as in Eq. (34),

$$\theta = \text{sgn}(\theta) \int_{\bar{a}}^r \frac{d\theta}{ds} ds = \text{sgn}(\theta) \int_{\bar{a}}^r \frac{\bar{a} \bar{b}}{s \sqrt{(s^2 - \bar{a}^2)(s^2 + \bar{b}^2)}} ds \quad (67)$$

$$= \text{sgn}(\theta) \tan^{-1} \left(\frac{\bar{b}}{\bar{a}} \sqrt{\frac{r^2 - \bar{a}^2}{r^2 + \bar{b}^2}} \right), \quad (68)$$

from which follows

$$r = \frac{\bar{a} \bar{b}}{\sqrt{\bar{b}^2 (\cos \theta)^2 - \bar{a}^2 (\sin \theta)^2}}. \quad (69)$$

This describes a trajectory inside \mathcal{B} that in euclidean geometry, where $x = r \cos \theta$ and $y = r \sin \theta$, would be an arc of a hyperbola centered on the center point \mathcal{C} of \mathcal{B} , with transverse (x) axis of length $2\bar{a}$ and conjugate (y) axis of length $2\bar{b}$. The initial conditions $r = R$ and $\theta = \delta$ at A produce from Eq. (69)

$$\bar{a} \bar{b} = R \sqrt{\bar{b}^2 (\cos \delta)^2 - \bar{a}^2 (\sin \delta)^2}, \quad (70)$$

which yields the constraint $\bar{b}/\bar{a} > \tan(\delta)$ on the axes of the hyperbola. Consequent on this equation and $\bar{a}^2 \bar{b}^2 = \bar{\beta}^2 - \bar{\alpha}^2 = R^3 h^2 / \lambda m$ is

$$h^2 = \frac{\lambda m}{R^3} \bar{a}^4 \frac{R^2 - r_0^2}{r_0^2 - \bar{a}^2}, \quad (71)$$

from which follow that $\bar{a} < r_0$ and $h \rightarrow \infty$ as $\bar{a} \rightarrow r_0$ (and vice versa).

From $d\theta/d\hat{\tau} = \dot{\theta} = -ch/r^2$ and Eq. (69) one can find the proper time elapsed on the particle's clock (if in fact there is a particle following the geodesic, and the particle has a clock, and $\hat{\tau}$ is the time measured by that clock) as follows:

$$\hat{\tau}_B - \hat{\tau}_A = \int_{\hat{\tau}_A}^{\hat{\tau}_B} d\hat{\tau} = -\frac{1}{ch} \int_{\delta}^{-\delta} r^2 d\theta = \frac{\bar{a}^2 \bar{b}^2}{ch} \int_{-\delta}^{\delta} \frac{1}{\bar{b}^2 (\cos \theta)^2 - \bar{a}^2 (\sin \theta)^2} d\theta \quad (72)$$

$$= \frac{2\bar{a}\bar{b}}{ch} \left[\tanh^{-1} \left(\frac{\bar{a}}{\bar{b}} \tan(\delta) \right) \right]. \quad (73)$$

To find the proper time elapsed on a clock at B synchronized with a clock at A requires computing $t_B - t_A$. A calculation like that leading up to Eq. (44) shows that

$$\left(\frac{dt}{d\theta} \right)^2 = \frac{\lambda^2 \bar{a}^2 \bar{b}^2 (h^2 - \bar{a}^2)(h^2 + \bar{b}^2)}{c^2 h^2 [h^2 - \bar{a}^2 (\sin \theta)^2 + \bar{b}^2 (\cos \theta)^2]^2}, \quad (74)$$

from which follows that $h \geq \bar{a}$ and

$$t_B - t_A = \pm \frac{2\lambda \bar{a}\bar{b}}{ch} \tan^{-1} \left(\sqrt{\frac{h^2 - \bar{a}^2}{h^2 + \bar{b}^2}} \tan(\delta) \right). \quad (75)$$

The geodesic with $h = \bar{a}$, on which no time t passes in the particle's trip from A to B, separates the geodesics on which the particle arrives before it started from those on which it arrives after it started. The flight time $T_{\epsilon=-1}$ of the particle as read on a clock at B perfectly synchronized with a matching clock at A is given by $T_{\epsilon=-1} = \sqrt{1 - f^2(R)} (t_B - t_A)$.

A calculation like that for Eq. (50) produces for the length $L_{\epsilon=-1}$ of the particle's path

$$L_{\epsilon=-1} = \frac{\bar{a}\bar{b}}{\lambda} \int_{-\delta}^{\delta} \sqrt{\frac{\bar{a}^4 (\sin \theta)^2 + (\lambda^2 - 1) [\bar{b}^2 (\cos \theta)^2 - \bar{a}^2 (\sin \theta)^2]^2 + \bar{b}^4 (\cos \theta)^2}{[\bar{b}^2 (\cos \theta)^2 - \bar{a}^2 (\sin \theta)^2]^3}} d\theta. \quad (76)$$

For the inertially measured flight time of the particle from A to B the analog of Eqs. (25) and (51) is

$$\bar{T}_{\epsilon=-1} := \bar{t}_B - \bar{t}_A = t_B - t_A - \frac{2}{c} \int_{\bar{a}}^R \frac{\sqrt{\lambda^2 - 1 - (\lambda m/R^3)r^2}}{1 + (\lambda m/R^3)r^2} dr. \quad (77)$$

III. DISCUSSION

Suppose a photon γ and a neutrino ν depart from a point A at time 0 and arrive at a point B at times T_γ and T_ν as measured by a clock at B perfectly synchronized with a matching clock at A. If $T_\nu < T_\gamma$, is one entitled to say that the neutrino traveled faster than the photon? One is not, for missing is any information about the lengths L_γ and L_ν of the paths that the particles followed. In the interpretation of the results of the experiment described in [6] it was assumed (in the absence of other, more realistic options) that $L_\gamma = L_\nu = d$, the euclidean distance from the neutrino source point A at CERN's European Laboratory for Particle Physics to a point B in the OPERA detector at LNGS (Laboratori Nazionale del Gran Sasso), and that $T_\gamma = d/c$. T_ν was reported to have been observed to be less than T_γ by about 57.8 ns.² This was interpreted to imply that the neutrino's speed exceeded that of light by about $(2.4 \times 10^{-5})c \approx 7.2$ km/sec. On its face this is not an allowable inference, as it compares the speed of the neutrino traveling through the gravitational field inside Earth to the speed of a photon traveling through empty space. Given, however, that the actual distance a photon would travel through a tunnel between A and B would likely differ very little from d , and that its speed in the tunnel should differ very little from c , the inference was not unreasonable.

A proper comparison between T_γ and T_ν must have the photon and the neutrino travel in the same space-time geometry, such as that inside the Earth as depicted in this paper. Even then the comparison cannot be exact, as the particles follow different routes from A to B, but in the applications detailed above for the CERN to Gran Sasso measurements the maximum separation between their routes is $(a - r_0)/\lambda = 1.00849 \times 10^{-23}$ cm and the neutrino's route is only 1.61152×10^{-33} cm longer than the photon's, so the comparison is nearly exact. The model predicted that the neutrino's flight time would exceed the photon's by 1.68745×10^{-23} cm/sec, whether measured by the

² Corrected at a later date to $T_\nu \approx T_\gamma - 6.5$ ns [8] and thereby brought into approximate compatibility with the results reported in [7].

clocks stationed at A and B or by clocks carried by free-falling inertial observers (allowed to penetrate Earth without interacting nongravitationally with its matter). As seen in Eqs. (26), (27), (60), and (61), these times would be greater than T_γ (the time a photon would take traveling in a vacuum) by about 10^{-15} sec as measured by the surface clocks, but less than T_γ by about 10^{-9} sec as measured by the inertial clocks. Moreover, as seen in Eq. (63), the average speeds of the neutrino and the photon as measured in the inertial frames along their paths, differing from one another by about 10^{-10} cm/sec, exceed c by about 321 meter/sec. This somewhat unintuitive result merely reflects the fact that the geometry inside Earth differs in a particular way from the geometry in a vacuum. The way it differs is determined here by the new, improved field equations employed to discover and govern it [1], field equations created to correct Einstein's unjustified assumption that inertial-passive mass (and therefore energy) can masquerade as active gravitational mass in the production of gravity [2].

Modeling Earth as a nonrotating, homogeneous, spherical ball in order to analyze photon and neutrino flights from CERN to Gran Sasso is, of course, dictated by the relative ease of solving the field equations (1) under those restrictions. Let us consider the possibility of relaxing those restrictions and what the effects of doing so might be.

- Allowing inhomogeneity while retaining the other restrictions could be accomplished by numerically solving the field equations with a radially varying density μ such as that profiled in [10] (based on [11]). Subsequent numerical computations of the various integrals in Secs. I and II would likely yield for $d = 731$ km results differing very little from those found here, inasmuch as the maximum depth of the photon's trajectory is $(R - r_0)/\lambda = (R - R \cos(\delta))/\lambda = 10.5$ km, and that of the neutrino's trajectory is less. For neutrino detectors contemplated as targets more distant from CERN, such as Majorana Demonstrator in South Dakota and Super-Kamiokande in Japan, the results might be significantly different from those for a constant density μ .
- To take into account Earth's rotation one would ideally find an interior solution of the field equations that matched up at the surface with some solution of the vacuum field equations that could reasonably be interpreted as modeling the gravitational field exterior to a rotating Earth. This would likely require giving up spherical symmetry in favor of an oblate spheroidal symmetry, which Earth has to a close approximation. If the exterior solution were taken to be a portion of the Kerr space-time manifold [12], finding a matching interior solution might be feasible with the density μ constant. Otherwise the problem would reduce to numerically solving partial differential equations in two variables, ρ and ϑ (or θ). The corrections to flight times of photons and neutrinos would likely be relatively small.

There are other variations to be taken into account, most notably the elevations above sea level of the starting and ending points of the photon and neutrino trajectories, and the mountains and valleys above the flight paths. These have been examined in detail in [10].

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